

REMOVAL OF PHASE TRANSITION OF THE CHEBYSHEV QUADRATIC AND THERMODYNAMICS OF HÉNON-LIKE MAPS NEAR THE FIRST BIFURCATION

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ABSTRACT. We treat a problem at the interface of dynamical systems and equilibrium statistical physics. It is well-known that the geometric pressure function

$$t \in \mathbb{R} \mapsto \sup_{\mu} \left\{ h_{\mu}(T_2) - t \int \log |dT_2(x)| d\mu(x) \right\}$$

of the Chebyshev quadratic map $T_2(x) = 1 - 2x^2$ ($x \in \mathbb{R}$) is not differentiable at $t = -1$. We show that this phase transition can be “removed”, by an arbitrarily small singular perturbation of the map T_2 into Hénon-like diffeomorphisms. A proof of this result relies on an elaboration of the well-known inducing techniques adapted to Hénon-like dynamics near the first bifurcation.

1. INTRODUCTION

The *thermodynamic formalism*, i.e., the formalism of equilibrium statistical physics developed by G. W. Gibbs and others, has been successfully brought into the ergodic theory of chaotic dynamical systems (see e.g., [3, 16] and the references therein). In the classical setting, it deals with a continuous map f of a compact metric space X and a continuous function φ on X , and looks for *equilibrium measures* which maximize (the minus of) the free energy $F_{\varphi}(\mu) = h_{\mu}(f) + \int \varphi d\mu$ among all f -invariant Borel probability measures on X . A relevant problem is to study the regularity of the *pressure function* $t \in \mathbb{R} \mapsto P(t\varphi)$, where $P(t\varphi) = \sup_{\mu} F_{t\varphi}(\mu)$.

The existence and uniqueness of equilibrium measures depends upon details of the system and the potential. For transitive uniformly hyperbolic systems and Hölder continuous potentials, the existence and uniqueness of equilibrium measures as well as the analyticity of the pressure function has been established in the pioneering works of Bowen, Ruelle and Sinai [3, 16, 19]. The latter property is interpreted as *the lack of phase transition*.

One important problem in dynamics is to understand structurally unstable, or nonhyperbolic systems [15]. The main problem which equilibrium statistical physics tries to clarify is that of phase transitions [16]. Hence, it is natural to study how phase transitions are affected by small perturbations of dynamics.

A natural candidate for a potential is the so-called *geometric potential* $\varphi(x) = -\log \|Df|E_x^u\|$, where E_x^u denotes the unstable direction at x which reflect the chaotic behavior of f . For nonhyperbolic systems, $x \mapsto E_x^u$ is often merely measurable, and may even be unbounded as in the case of one-dimensional maps with critical points. These defects sometimes lead to the

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occurrence of phase transitions, e.g., the loss of analyticity or differentiability of the pressure function. Typically, at phase transitions, there exist multiple equilibrium measures.

As an emblematic example, consider the family of quadratic maps $T_a: x \in \mathbb{R} \mapsto 1 - ax^2$ ($a > -1/4$) and the associated family of geometric pressure functions $t \in \mathbb{R} \mapsto P(-t \log |dT_a|)$ given by

$$(1) \quad P(-t \log |dT_a|) = \sup_{\mu} \left\{ h_{\mu}(T_a) - t \int \log |dT_a| d\mu \right\}.$$

Here, $h_{\mu}(T_a)$ denotes the Kolmogorov-Sinai entropy of (T_a, μ) and the supremum is taken over all T_a -invariant Borel probability measures.

For $a > 2$, the Julia set does not contain the critical point $x = 0$, and so the dynamics is uniformly hyperbolic and structurally stable. According to the classical theory, for any $t \in \mathbb{R}$ there exists a unique equilibrium measure for the potential $-t \log |dT_a|$, and the geometric pressure function is real analytic. At the first bifurcation parameter $a = 2$ the Julia set contains the critical point, and so the dynamics is nonhyperbolic and structurally unstable. The Lyapunov exponent of any ergodic measure is either $\log 2$ or $\log 4$, and it is $\log 4$ only for the Dirac measure, denoted by δ_{-1} , at the orientation-preserving fixed point. Equilibrium measures for the potential $-t \log |dT_2|$ are: (i) δ_{-1} if $t < -1$; (ii) δ_{-1} and μ_{ac} if $t = -1$; (iii) μ_{ac} if $t > -1$, where μ_{ac} denotes the absolutely continuous invariant probability measure. Correspondingly, the pressure function is not real analytic at $t = -1$:

$$P(-t \log |dT_2|) = \begin{cases} -t \log 4 & \text{if } t \leq -1; \\ (1 - t) \log 2 & \text{if } t > -1. \end{cases}$$

We say T_2 displays the *freezing phase transition in negative spectrum*, to be defined below (see the paragraph just before the Main Theorem).

This phase transition is due to the fact that the measure δ_{-1} is anomalous: it has the maximal Lyapunov exponent, and this value is isolated in the set of Lyapunov exponents of all ergodic measures. For all $a \in (-1/4, 2)$ the Dirac measure at the orientation preserving fixed point continues to be anomalous, and therefore all the quadratic maps continue to display the freezing phase transition [8, Proposition 4]. The freezing phase transition in negative spectrum is often caused by anomalous periodic points. For example, see [11] for results on certain two-dimensional real polynomial endomorphism, and [12] for a complete characterization on rational maps of degree ≥ 2 on the Riemannian sphere.

An elementary observation is that any nonhyperbolic one-dimensional map sufficiently close to T_2 in the C^2 -topology displays the freezing phase transition in negative spectrum. This raises the following question: is it possible to remove the phase transition of T_2 by an arbitrarily small singular perturbation to higher dimensional nonhyperbolic maps? More precisely we ask:

(Removability problem) *Is it possible to “approximate” $T_2: x \in \mathbb{R} \mapsto 1 - 2x^2$ by higher dimensional nonhyperbolic maps which do not display the freezing phase transition in negative spectrum?*

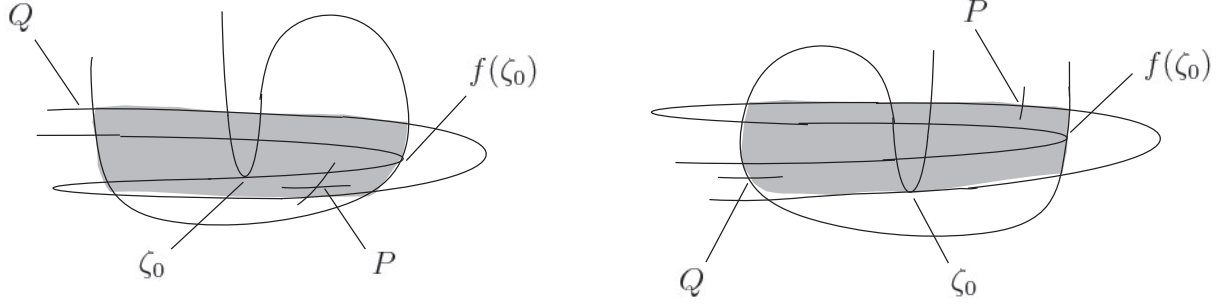


FIGURE 1. Organization of the invariant manifolds at $a = a^*$. There exist two fixed saddles P, Q near $(1/2, 0), (-1, 0)$ respectively. In the case $\det Df > 0$ (left), $W^s(Q)$ meets $W^u(Q)$ tangentially. In the case $\det Df < 0$ (right), $W^s(Q)$ meets $W^u(P)$ tangentially. The shaded regions represent the rectangle R (See Sect.2.1).

The aim of this paper is to show that the phase transition of T_2 can be removed, by an arbitrarily small singular perturbation along *the first bifurcation curve* of a family of Hénon-like diffeomorphisms

$$f_a: (x, y) \in \mathbb{R}^2 \mapsto (1 - ax^2, 0) + b \cdot \Phi(a, b, x, y), \quad a \in \mathbb{R}, \quad 0 < b \ll 1,$$

where a is near 2, Φ is bounded continuous in a, b, x, y and C^2 in a, x, y . The parameter a controls the nonlinearity, and the b controls the dissipation of the map. Note that, with $b = 0$ the family degenerates into the family of quadratic maps.

We proceed to recall some known facts on the first bifurcation of the family of Hénon-like diffeomorphisms. If there is no fear of confusion, we suppress a from notation and write f for f_a , and so on. For (a, b) near $(2, 0)$ let P, Q denote the fixed saddles of f near $(1/2, 0)$ and $(-1, 0)$ respectively. The stable and unstable manifolds of P are respectively defined as follows:

$$W^s(P) = \{z \in \mathbb{R}^2: f^n(z) \rightarrow P \text{ as } n \rightarrow +\infty\};$$

$$W^u(P) = \{z \in \mathbb{R}^2: f^n(z) \rightarrow P \text{ as } n \rightarrow -\infty\}.$$

The stable and unstable manifolds of Q are defined in the same way. It is known [1, 6, 7, 22] that there is a *first bifurcation parameter* $a^* = a^*(b) \in \mathbb{R}$ with the following properties:

- if $a > a^*$, then the non wandering set is a uniformly hyperbolic horseshoe;
- if $a = a^*$, then there is a single orbit of homoclinic or heteroclinic tangency involving (one of) the two fixed saddles (see FIGURE 1). In the case $\det Df > 0$ (orientation preserving), $W^s(Q)$ meets $W^u(Q)$ tangentially. In the case $\det Df < 0$ (orientation reversing), $W^s(Q)$ meets $W^u(P)$ tangentially. The tangency is quadratic, and the one-parameter family $\{f_a\}_{a \in \mathbb{R}}$ unfolds the tangency at a^* generically. An incredibly rich array of dynamical complexities is unleashed in the unfolding of this tangency (see e.g., [15] and the references therein);
- $a^* \rightarrow 2$ as $b \rightarrow 0$.

The curve $\{(a^*(b), b) \in \mathbb{R}: b > 0\}$ is a *nonhyperbolic path* to the quadratic map T_2 , consisting of parameters corresponding to nonhyperbolic dynamics. The main theorem claims that $f_{a^*(b), b}$ does *not* display the freezing phase transition in negative spectrum.

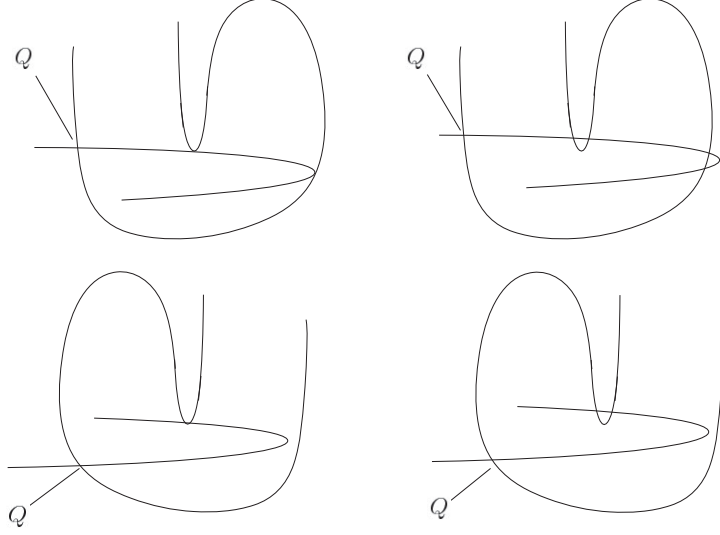


FIGURE 2. Organization of $W^s(Q)$ and $W^u(Q)$: $\det Df > 0$ and $a > a^{**}$ close to a^{**} (upper-right); $\det Df > 0$ and $a = a^{**}$ (upper-left); $\det Df < 0$ and $a > a^{**}$ close to a^{**} (lower-right); $\det Df < 0$ and $a = a^{**}$ (lower-left).

To give a precise statement of result we need a preliminary discussion. We first make explicit the range of the parameter a to consider. Assume $0 < b \ll 1$. Let $W_{\text{loc}}^s(Q)$ denote the compact curve in $W^s(Q)$ containing Q such that $W_{\text{loc}}^s(Q) \setminus \{Q\}$ has two connected components of length \sqrt{b} . Let $\psi: \mathbb{R} \rightarrow W^u(Q)$ denote the isometric embedding such that $\psi(0) = Q$ and $\psi(\{x \in \mathbb{R}: x < 0\}) \cap \Omega = \emptyset$. Let

$$\ell^u = \begin{cases} \psi(1 - 1/100, 1 + 1/100) & \text{if } \det Df > 0; \\ \psi(3 - 1/100, 3 + 1/100) & \text{if } \det Df < 0. \end{cases}$$

Define

$$\mathcal{G} = \{a \in \mathbb{R}: f^{-2}(W_{\text{loc}}^s(Q)) \text{ and } \ell^u \text{ bounds a compact domain}\},$$

and

$$a^{**} = \inf \mathcal{G}.$$

Note that $a^* \in \mathcal{G}$, $a^{**} < a^*$, $a^{**} \rightarrow 2$ as $b \rightarrow 0$, and that at $a = a^{**}$, $f^{-2}(W_{\text{loc}}^s(Q))$ is tangent to ℓ^u quadratically. Since the family $\{f_a\}_a$ unfolds the tangency ζ_0 at $a = a^*$ generically, $(a^{**}, a^*] \subset \mathcal{G}$. In this paper we assume $a \in (a^{**}, a^*]$.

Let Ω denote the non wandering set of f , which is a compact f -invariant set. For nonhyperbolic dynamics beyond the parameter a^* , the notion of “unstable direction” is not clear. In the next paragraph, we circumvent this point with the Pesin theory (See e.g., [9]), by introducing a Borel set Λ on which an *unstable direction* E^u makes sense.

Given $\chi > \epsilon > 0$, for each integer $k \geq 1$ define $\Lambda_k(\chi, \epsilon)$ to be the set of points $z \in \Omega$ for which there is a one-dimensional subspace E_z^u of $T_z \mathbb{R}^2$ such that for every integers $m \in \mathbb{Z}$, $n \geq 1$ and for all vectors $v^u \in Df^m(E_z^u)$,

$$\|D_{f^m(z)} f^{-n}(v^u)\| \leq e^{\epsilon k} e^{-(\chi - \epsilon)n} e^{\epsilon |m|} \|v^u\|.$$

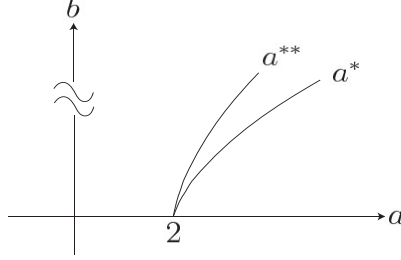


FIGURE 3. The landscape in (a, b) -space, $b \ll 1$. The parameters $a^* = a^*(b)$ and $a^{**} = a^{**}(b)$ converge to 2 as $b \rightarrow 0$. The dynamics for parameters at the right of the a^* -curve is uniformly hyperbolic.

Since f^{-1} expands area, the subspace E_z^u with this property is unique when it exists, and characterized by the following backward contraction property

$$(2) \quad \limsup_{n \rightarrow +\infty} \frac{1}{n} \log \|Df^{-n}|E_z^u\| < 0.$$

Here, $\|\cdot\|$ denotes the norm induced from the Euclidean metric on \mathbb{R}^2 . Note that $\Lambda_k(\chi, \epsilon)$ is a closed set, and $z \in \Lambda_k(\chi, \epsilon) \mapsto E_z^u$ is continuous. Moreover, if $z \in \Lambda_k(\chi, \epsilon)$ then $f(z), f^{-1}(z) \in \Lambda_{k+1}(\chi, \epsilon)$. Therefore, the Borel set

$$\Lambda(\chi, \epsilon) = \bigcup_{k=1}^{\infty} \Lambda_k(\chi, \epsilon)$$

is f -invariant: $f(\Lambda(\chi, \epsilon)) = \Lambda(\chi, \epsilon)$. Then the Borel set

$$\Lambda = \bigcup_{\epsilon > 0} \bigcup_{\chi > \epsilon} \Lambda(\chi, \epsilon)$$

is f -invariant as well, and the map $z \in \Lambda \mapsto E_z^u$ is Borel measurable with the invariance property $Df(E_z^u) = E_{f(z)}^u$. The one-parameter family of potentials we are concerned with is

$$-t \log J^u \quad t \in \mathbb{R},$$

where $J^u(z) = \|Df|E_z^u\|$ ($z \in \Lambda$). Since Ω is compact and f is a diffeomorphism, J^u is bounded from above and bounded away from zero. We shall only take into consideration measures which give full weight to Λ .

The chaotic behavior of f is produced by the non-uniform expansion along the unstable direction E^u , and thus a good deal of information will be obtained by studying the associated geometric pressure function $t \in \mathbb{R} \mapsto P(-t \log J^u)$ defined by

$$(3) \quad P(-t \log J^u) = \sup \{h_\mu(f) - t\lambda^u(\mu) : \mu \in \mathcal{M}_0(f)\},$$

where

$$\mathcal{M}_0(f) = \{\mu \in \mathcal{M}(f) : \mu(\Lambda) = 1\},$$

and $h_\mu(f)$ denotes the entropy of (f, μ) , and

$$\lambda^u(\mu) = \int \log J^u d\mu,$$

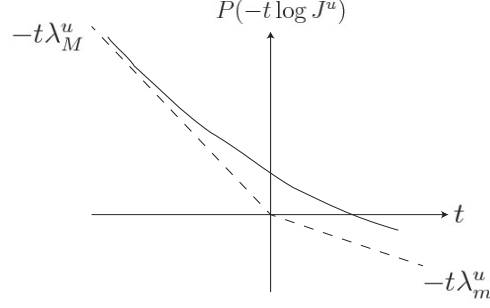


FIGURE 4. At $a = a^*$, the graph of the pressure function $t \mapsto P(-t \log J^u)$ has the line $-t\lambda_M^u$ as its asymptote as $t \rightarrow -\infty$, but never touches it (Main Theorem).

which we call an *unstable Lyapunov exponent* of $\mu \in \mathcal{M}_0(f)$. Let us call any measure in $\mathcal{M}(f)$ which attains the supremum in $P(-t \log J^u)$ an *equilibrium measure* for the potential $-t \log J^u$.

We suggest the reader to compare (1) and (3). One important difference is that the function $\log |dT_a|$ in (1) is unbounded, while the function $\log J^u$ in (3) is bounded. Another important difference is that the class of measures taken into consideration is reduced in (3).

It is natural to ask in which case $\mathcal{M}_0(f) = \mathcal{M}(f)$. This is the case for $a = a^*$ because $\Lambda = \Omega$ from the result in [17]. In fact, $\mathcal{M}_0(f) = \mathcal{M}(f)$ still holds for “most” parameters immediately right after the first bifurcation at a^* . See Sect.4.2 for more details.

The potential $-t \log J^u$ and the associated pressure function deserve to be called *geometric*, primarily because Bowen’s formula holds at $a = a^*$ [18, Theorem B]: the equation $P(-t \log J^u) = 0$ has the unique solution which coincides with the (unstable) Hausdorff dimension of Ω . We expect that the same formula holds for the above “most” parameters.

We are in position to state our main result. Let

$$\begin{aligned}\lambda_m^u &= \inf\{\lambda^u(\mu) : \mu \in \mathcal{M}_0(f)\}; \\ \lambda_M^u &= \sup\{\lambda^u(\mu) : \mu \in \mathcal{M}_0(f)\},\end{aligned}$$

and define *freezing points* t_c, t_f by

$$\begin{aligned}t_c &= \inf\{t \in \mathbb{R} : P(-t \log J^u) > -t\lambda_M^u\}; \\ t_f &= \sup\{t \in \mathbb{R} : P(-t \log J^u) > -t\lambda_m^u\}.\end{aligned}$$

This denomination is because equilibrium measures do not change any more for $t < t_c$ or $t > t_f$. Indeed it is elementary to show the following:

- $-\infty \leq t_c < 0 < t_f \leq +\infty$;
- if $t \in (t_c, t_f)$, then any equilibrium measure for $-t \log J^u$ (if it exists) has positive entropy;
- if $t \leq t_c$, then $P(-t \log J^u) = -t\lambda_M^u$. If $t \geq t_f$, then $P(-t \log J^u) = -t\lambda_m^u$.

Let us say that f displays the freezing phase transition in negative (resp. positive) spectrum if t_c (resp. t_f) is finite.

Main Theorem. *Let $\{f_a\}$ be a family of Hénon-like diffeomorphisms. If $b > 0$ is sufficiently small and $a \in (a^{**}(b), a^*(b))$, then f_a does not display the freezing phase transition in negative spectrum. If $a = a^*(b)$, then $P(-t \log J^u) = -t\lambda_M^u + o(1)$ as $t \rightarrow -\infty$.*

The main theorem states that the graph of the pressure function does not touch the line $-t\lambda_M^u$. At $a = a^*$ we have more information: this line is the asymptote of the graph of $P(-t \log J^u)$ as $t \rightarrow -\infty$ (see FIGURE 4).

The main theorem reveals a difference between the bifurcation structure of quadratic maps and that of Hénon-like maps from the thermodynamic point of view. As mentioned earlier, the quadratic maps display the freezing phase transition in negative spectrum for all parameters beyond the bifurcation, while this is not the case for Hénon-like maps.

The freezing phase transition for negative spectrum does occur for some parameters $< a^{**}$. It is well-known that there exists a parameter set of positive Lebesgue measure corresponding to non-uniformly hyperbolic strange attractors [2, 13, 24]. For these parameters, the non-wandering set is the disjoint union of the strange attractor and the fixed saddle near $(-1, 0)$ [4, 5]. For these parameters it is possible to show that the Dirac measure at the saddle is anomalous.

Regarding freezing phase transitions in positive spectrum of Hénon-like maps, the known result is very much limited. Let δ_Q denote the Dirac measure at Q . It was proved in [23, Proposition 3.5(b)] that if $a = a^*$ and $\lambda_m^u = (1/2)\lambda^u(\delta_Q)$, then f does not display the freezing phase transition in positive spectrum. However, since $\lambda_m^u \rightarrow \log 2$ and $\lambda^u(\delta_Q) \rightarrow \log 4$ as $b \rightarrow 0$, it is not easy to prove or disprove this equality.

For a proof of the main theorem we first show that δ_Q is the unique measure which maximizes the unstable Lyapunov exponent (see Lemma 2.4). Then it suffices to show that for any $t < 0$ there exists a measure $\nu_t \in \mathcal{M}_0(f)$ such that $h_{\nu_t}(f) - t\lambda^u(\nu_t) > -t\lambda^u(\delta_Q)$. To see the subtlety of showing this, note that from the variational principle ν_t must satisfy

$$(4) \quad t(\lambda^u(\nu_t) - \lambda^u(\delta_Q)) < h_{\nu_t}(f) \leq h_{\text{top}}(f),$$

where $h_{\text{top}}(f)$ denotes the topological entropy of f . As $-t$ becomes large, the unstable Lyapunov exponent becomes more important and we must have $\lambda^u(\nu_t) \rightarrow \lambda^u(\delta_Q)$ as $t \rightarrow -\infty$. A naive application of the Poincaré-Birkhoff-Smale theorem [15] to a transverse homoclinic point of Q indeed yields a measure whose unstable Lyapunov exponent is approximately that of δ_Q , but it is not clear if the entropy is sufficiently large for the first inequality in (4) to hold.

Our approach is based on the well-known inducing techniques adapted to the Hénon-like maps, inspired by Makarov & Smirnov [12] (see also Leplaideur [10]). The idea is to carefully choose for each $t < 0$ a hyperbolic subset H_t of Ω such that the first return map to it is topologically conjugate to the full shift on a finite number of symbols. We then spread out the maximal entropy measure of the first return map to produce a measure with the desired properties. As $-t$ becomes large, more symbols are needed in order to fulfill the first inequality in (4).

The hyperbolic set H_t is chosen in such a way that any orbit contained in it spends a very large proportion of time near the saddle Q , during which the unstable directions are roughly parallel to E_Q^u . More precisely, for any point $z \in H_t$ with the first return time $R(z)$ to H_t , the fraction

$$\frac{1}{R(z)} \# \{n \in \{0, 1, \dots, R(z) - 1\} : |f^n(z) - Q| \ll 1 \text{ and } \text{angle}(E_{f^n(z)}^u, E_Q^u) \ll 1\}$$

is nearly 1. A standard bounded distortion argument then allows us to copy the unstable Lyapunov exponent of δ_Q . Note that, if the unstable direction is not continuous (which is indeed the case at $a = a^*$ [18] and considered to be the case for most $a < a^*$), then the closeness of base points $|f^n(z) - Q| \ll 1$ does not guarantee the closeness of the corresponding unstable directions $\text{angle}(E_{f^n(z)}^u, E_Q^u) \ll 1$.

In order to let points stay near the saddle Q for a very long period of time, one must allow them to enter deeply into the critical zone. As a price to pay, the directions of E^u along the orbits get switched due to the folding behavior near the critical zone. In order to restore the horizontality of the direction and establish the closeness to E_Q^u , we develop the binding argument relative to dynamically critical points, inspired by Benedicks & Carleson [2]. The point is that one can choose the hyperbolic set H_t so that the effect of the folding is not significant, and the restoration can be done in a uniformly bounded time. This argument works at the first bifurcation parameter a^* , and even for all parameters in (a^{**}, a^*) because only those parts in the phase space not being destroyed by the homoclinic bifurcation are involved.

The rest of this paper consists of three sections. In Sect.2 we introduce the key concept of critical points, and develop estimates related to them. In Sect.3 we use the results in Sect.2 to construct the above-mentioned hyperbolic set. In Sect.4 we finish the proof of the main theorem and provide more details, on the abundance of parameters satisfying $\mathcal{M}_0(f) = \mathcal{M}(f)$.

2. LOCAL ANALYSIS NEAR CRITICAL ORBITS

For the rest of this paper, we assume $f = f_a$ and $a \in (a^{**}, a^*]$. In this section we develop a local analysis near the orbits of critical points. The main result is Proposition 2.7 which controls the norms of the derivatives in the unstable direction, along orbits which pass through critical points.

For the rest of this paper we are concerned with the following positive small constants: τ , δ , b chosen in this order, the purposes of which are as follows:

- τ is used to exclusively in the proof of Proposition 2.7;
- δ determines the size of a critical region (See Sect.2.5);
- b determines the magnitude of the reminder term in (1).

We shall write C with or without indices to denote any constant which is independent of τ , δ , b . For $A, B > 0$ we write $A \approx B$ if both A/B and B/A are bounded from above by constants independent of τ , δ , b . If $A \approx B$ and the constants can be made arbitrarily close to 1 by appropriately choosing τ , δ , b , then we write $A \asymp B$.

For a nonzero tangent vector $v = \begin{pmatrix} \xi \\ \eta \end{pmatrix}$ at a point $z \in \mathbb{R}^2$, define $\text{slope}(v) = |\eta|/|\xi|$ if $\xi \neq 0$, and $\text{slope}(v) = \infty$ if $\xi = 0$. Similarly, for the one-dimensional subspace V of $T_z\mathbb{R}^2$ spanned by v , define $\text{slope}(V) = \text{slope}(v)$. Given a C^1 curve γ in \mathbb{R}^2 , the length is denoted by $\text{length}(\gamma)$. The tangent space of γ at $z \in \gamma$ is denoted by $T_z\gamma$. The Euclidean distance between two points z_1, z_2 of \mathbb{R}^2 is denoted by $|z_1 - z_2|$. The angle between two tangent vectors v_1, v_2 is denoted by $\text{angle}(v_1, v_2)$. The interior of a subset X of \mathbb{R}^2 is denoted by $\text{Int}(X)$.

2.1. The non wandering set. Recall that the map f has exactly two fixed points, which are saddles: P is the one near $(1/2, 0)$ and Q is the other one near $(-1, 0)$. The orbit of tangency at the first bifurcation parameter $a = a^*$ intersects a small neighborhood of the origin $(0, 0)$

exactly at one point, denoted by ζ_0 . If $\det Df > 0$ then $\zeta_0 \in W^s(Q) \cap W^u(Q)$. If $\det Df < 0$ then $\zeta_0 \in W^s(Q) \cap W^u(P)$ (See FIGURE 1).

By a *rectangle* we mean any compact domain bordered by two compact curves in $W^u(P) \cup W^u(Q)$ and two in $W^s(P) \cup W^s(Q)$. By an *unstable side* of a rectangle we mean any of the two boundary curves in $W^u(P) \cup W^u(Q)$. A *stable side* is defined similarly.

In the case $\det Df > 0$ (resp. $\det Df < 0$) let $R = R_a$ denote the rectangle which is bordered by two compact curves in $W^u(Q)$ (resp. $W^u(P)$) and two in $W^s(Q)$, and contains Ω . The rectangle with these properties is unique, and is located near the segment $\{(x, 0) \in \mathbb{R}^2: |x| \leq 1\}$. One of the stable sides of R contains Q , which is denoted by α_0^- . The other stable side of R is denoted by α_0^+ . We have $f(\alpha_0^+) \subset \alpha_0^-$. At $a = a^*$, one of the unstable sides of R contains the point ζ_0 of tangency near $(0, 0)$ (See FIGURE 1).

2.2. Non critical behavior. Define

$$I(\delta) = \{(x, y) \in R: |x| < \delta\},$$

and call it a *critical region*.

By a $C^2(b)$ -*curve* we mean a compact, nearly horizontal C^2 curve in R such that the slopes of tangent vectors to it are $\leq \sqrt{b}$ and the curvature is everywhere $\leq \sqrt{b}$.

Lemma 2.1. *Let γ be a $C^2(b)$ -curve in $R \setminus I(\delta)$. Then $f(\gamma)$ is a $C^2(b)$ -curve.*

Proof. Follows from Lemma 2.2 and the lemma below. \square

Put $\lambda_0 = \frac{99}{100} \log 2$.

Lemma 2.2. *Let $z \in R$ and $n \geq 1$ be an integer such that $z, f(z), \dots, f^{n-1}(z) \notin I(\delta)$. Then for any nonzero vector v at z with $\text{slope}(v) \leq \sqrt{b}$,*

$$\text{slope}(Df^n(v)) \leq \sqrt{b} \quad \text{and} \quad \|Df^n(v)\| \geq \delta e^{\lambda_0 n} \|v\|.$$

If moreover $f^n(z) \in I(\delta)$, then $\|Df^n(v)\| \geq e^{\lambda_0 n} \|v\|$.

Proof. Follows from the fact that $|DT_2(x)| > 2$ outside of $[-1, 1]$ and that b is very small. \square

Lemma 2.3. ([20, Lemma 2.3]) *Let γ be a C^2 curve in R and $z \in \gamma$. For each integer $i \geq 0$ let $\kappa_i(z)$ denote the curvature of $f^i(\gamma)$ at $f^i(z)$. Then for any nonzero vector v tangent to γ at z ,*

$$\kappa_i(z) \leq (Cb)^i \frac{\|v\|^3}{\|Df^i(v)\|^3} \kappa_0(z) + \sum_{j=1}^i (Cb)^j \frac{\|Df^j(v)\|^3}{\|Df^i(v)\|^3}.$$

2.3. Lyapunov maximizing measure. Recall that $\mathcal{M}_0(f)$ is the set of f -invariant Borel probability measures which gives total mass to the set Λ . The next lemma states that δ_Q is the unique measure which maximizes the unstable Lyapunov exponent among measures in $\mathcal{M}_0(f)$.

Lemma 2.4. *For any $\mu \in \mathcal{M}_0(f) \setminus \{\delta_Q\}$, $\lambda^u(\mu) < \lambda^u(\delta_Q)$.*

Proof. From the linearity of the unstable Lyapunov exponent as a function of measures, it suffices to consider the case where μ is ergodic. Let

$$\text{supp}(\mu) = \bigcap \{F: F \text{ is a closed subset of } \Omega \text{ and } \mu(F) = 1\}.$$

Reducing $b > 0$ if necessary, one can show that $\lambda^u(\mu) < \lambda^u(\delta_Q)$ holds for any ergodic μ with $\text{supp}(\mu) \cap I(\delta) = \emptyset$. In the case $\text{supp}(\mu) \cap I(\delta) \neq \emptyset$ we have $\mu(I(\delta)) > 0$. From the Ergodic Theorem, it is possible to take a point $z \in I(\delta)$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=0}^{n-1} \log \|Df|E_{f^i(z)}^u\| = \lambda^u(\mu) > 0,$$

and

$$\lim_{n \rightarrow \infty} \frac{1}{n} \#\{i \in \{0, 1, \dots, n-1\} : f^i(z) \in I(\delta)\} = \mu(I(\delta)) > 0.$$

Define a sequence $m_1 \leq m_1 + r_1 \leq m_2 \leq m_2 + r_2 \leq m_3 \leq \dots$ of nonnegative integers inductively as follows. Start with $m_1 = 0$. Let $k \geq 1$ and m_k be such that $f^{m_k}(z) \in I(\delta)$. If $\text{slope}(E_{f^{m_k+1}(z)}^u) \leq 1/b^{1/3}$, then define

$$r_k = 0 \quad \text{and} \quad m_{k+1} = \min\{n > m_k : f^n(z) \in I(\delta)\}.$$

If $\text{slope}(E_{f^{m_k+1}(z)}^u) > 1/b^{1/3}$, then define

$$r_k = \min\{i > 1 : \text{slope}(E_{f^{m_k+i}(z)}^u) \leq 1/b^{1/3}\} \quad \text{and} \quad m_{k+1} = \min\{n \geq m_k + r_k : f^n(z) \in I(\delta)\}.$$

Since $\lambda^u(\mu) > 0$, $r_k < \infty$.

The form of our map (1) gives $\text{slope}(E_{f^n(z)}^u) \leq 1/b^{1/3}$ for every $n \in \{m_k + r_k, m_k + r_k + 1, \dots, m_{k+1}\}$. This implies

$$\|Df^{m_{k+1}-m_k-r_k}|E_{f^{m_k+r_k}(z)}^u\| \leq 2e^{\lambda^u(\delta_Q)(m_{k+1}-m_k-r_k)}.$$

If $\text{slope}(E_{f^n(z)}^u) \geq 1/b^{1/3}$ then $\|Df^{f^n(z)}f|E_{f^n(z)}^u\| \leq \sqrt{b}$. Hence

$$\begin{aligned} (5) \quad \|Df^{m_{k+1}-m_k}|E_{f^{m_k}(z)}^u\| &\leq 2e^{\lambda^u(\delta_Q)(m_{k+1}-m_k-r_k)} \min\{b^{\frac{r_k}{2}}, 3\delta\} \\ &\leq e^{\lambda^u(\delta_Q)(m_{k+1}-m_k)} \min\{b^{\frac{r_k}{3}}, 3\delta\}. \end{aligned}$$

For each integer $n \geq 0$ define $V_n = \{i \in \{0, 1, \dots, n-1\} : \text{slope}(E_{f^i(z)}^u) \geq 1/b^{1/3}\}$. If $\limsup_{n \rightarrow \infty} \#V_n/n > 0$, then the first alternative in (5) yields

$$\begin{aligned} \frac{1}{m_{k+1}} \log \|Df^{m_{k+1}}|E_z^u\| &\leq \lambda^u(\delta_Q) + \frac{1}{3} \log b \cdot \frac{1}{m_{k+1}} \sum_{i=1}^k r_i \\ &= \lambda^u(\delta_Q) + \frac{1}{3} \log b \cdot \frac{1}{m_{k+1}} \#V_{m_{k+1}}. \end{aligned}$$

Taking the upper limit as $k \rightarrow \infty$ yields $\lambda^u(\mu) < \lambda^u(\delta_Q)$. If $\limsup_{n \rightarrow \infty} \#V_n/n = 0$, then note that $k = \{n \in \{0, 1, \dots, m_{k+1}-1\} : f^n(z) \in I(\delta) \text{ and } \text{slope}(E_{f^n(z)}^u) \leq 1/b^{1/3}\}$ and $k/m_{k+1} \rightarrow \mu(I(\delta)) > 0$ as $k \rightarrow \infty$. Then the second alternative in (5) yields

$$\frac{1}{m_{k+1}} \log \|Df^{m_{k+1}}|E_z^u\| \leq \lambda^u(\delta_Q) + \log \delta \cdot \frac{k}{m_{k+1}}.$$

Taking the upper limit as $k \rightarrow \infty$ yields $\lambda^u(\mu) < \lambda^u(\delta_Q)$. □

2.4. C^1 -closeness due to disjointness. Corollary 2.6 below states that the pointwise convergence of pairwise disjoint $C^2(b)$ -curves implies the C^1 -convergence. This fact was already used in the precious works for Hénon-like maps, e.g., [17, 18]. We include precise statements and proofs for the reader's convenience.

Lemma 2.5. *Let $\varepsilon \in (0, (1 + \sqrt{b})^{-2})$ and let γ_1, γ_2 be two disjoint $C^2(b)$ -curves parametrized by arc length such that:*

- (i) $\gamma_1(s), \gamma_2(s)$ are defined for $s \in [-\varepsilon, \varepsilon]$;
- (ii) $|\gamma_1(0) - \gamma_2(0)| \leq \varepsilon^2$.

Then the following holds:

- (a) $\text{angle}(T_{\gamma_1(0)}\gamma_1, T_{\gamma_2(0)}\gamma_2) \leq \sqrt{\varepsilon}$;
- (b) $|\gamma_1(s) - \gamma_2(s)| \leq 2\varepsilon^{\frac{3}{2}}$ for all $s \in [-\varepsilon, \varepsilon]$.

Proof. Write $L(s) = \gamma_1(s) - \gamma_2(s)$. By the mean value theorem, for any t in between 0 and s there exists $\theta(t)$ in between 0 and t such that $\dot{L}(t) = \dot{L}(0) + \ddot{L}(\theta(t))t$, where the dot “ \cdot ” denotes the t -derivative. Integrating this equality gives

$$(6) \quad L(s) = L(0) + \int_0^s \dot{L}(t)dt = L(0) + \dot{L}(0)s + \int_0^s \ddot{L}(\theta(t))tdt.$$

We argue by contradiction assuming $\|\dot{L}(0)\| > (1/2)\sqrt{\varepsilon}$. The assumption $\varepsilon < (1 + \sqrt{b})^{-2}$, (ii) and $|\ddot{L}| \leq 2\sqrt{b}$ give

$$(7) \quad \left| L(0) + \int_0^\varepsilon \ddot{L}(\theta(t))tdt \right| \leq |L(0)| + 2\sqrt{b}\varepsilon^2 \leq \varepsilon^2 + 2\sqrt{b}\varepsilon^2 < \|\dot{L}(0)\|\varepsilon.$$

A comparison of (6) with (7) shows that the sign of $L(\varepsilon)$ coincides with that of $\|\dot{L}(0)\|\varepsilon$. The same argument shows that the sign of $L(-\varepsilon)$ coincides with that of $-\|\dot{L}(0)\|\varepsilon$. From the intermediate value theorem it follows that $L(s) = 0$ for some s , namely γ_1 intersects γ_2 , a contradiction. Hence $\text{angle}(T_{\gamma_1(0)}\gamma_1, T_{\gamma_2(0)}\gamma_2) < 2\|\dot{L}(0)\| \leq \sqrt{\varepsilon}$ and (a) holds. (b) follows from (6), (ii), (a) and $|\ddot{L}| \leq 2\sqrt{b}$. \square

Corollary 2.6. *Let $\{\gamma_n\}_{n=0}^{+\infty}$ be a sequence of pairwise disjoint $C^2(b)$ -curves which as a sequence of C^2 functions converges pointwise to a function γ as $n \rightarrow +\infty$. Then the graph of γ is a C^1 -curve and the slopes of its tangent directions are everywhere $\leq \sqrt{b}$.*

Proof. From Lemma 2.5(b), the pointwise convergence implies the uniform C^0 convergence. From Lemma 2.5(a), the uniform C^1 convergence follows. \square

2.5. Critical points. From the hyperbolicity of the fixed saddle Q , there exist mutually disjoint connected open sets U^- , U^+ independent of b such that $\alpha_0^- \subset U^-$, $\alpha_0^+ \subset U^+$, $U^+ \cap f(U^+) = \emptyset = U^+ \cap f(U^-)$ and a foliation \mathcal{F}^s of $U = U^- \cup U^+$ by one-dimensional vertical leaves such that:

- (a) $\mathcal{F}^s(Q)$, the leaf of \mathcal{F}^s containing Q , contains α_0^- ;
- (b) if $z, f(z) \in U$, then $f(\mathcal{F}^s(z)) \subset \mathcal{F}^s(f(z))$;
- (c) let $e^s(z)$ denote the unit vector in $T_z\mathcal{F}^s(z)$ whose second component is positive. Then $z \mapsto e^s(z)$ is C^1 , $\|D_z f e^s(z)\| \leq Cb$ and $\|D_z e^s(z)\| \leq C$;
- (d) if $z, f(z) \in U$, then $\text{slope}(e^s(z)) \geq C/\sqrt{b}$.

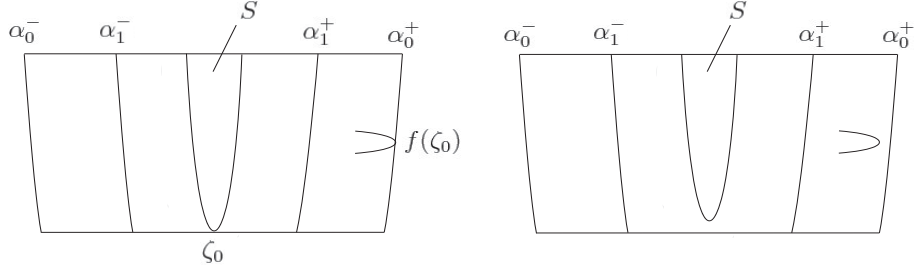


FIGURE 5. The lenticular domain S : $a = a^*$ (left); $a < a^*$ (right).

Let γ be a $C^2(b)$ -curve in $I(\delta)$. We say $\zeta \in \gamma$ is a *critical point on γ* if $\zeta \in S$ and $f(\gamma)$ is tangent to $\mathcal{F}^s(f(\zeta))$. If ζ is a critical point on a $C^2(b)$ -curve γ , then we say γ *admits ζ* . For simplicity, we sometimes refer to ζ as a *critical point* without referring to γ .

Let ζ be a critical point. Note that $f(\zeta) \in U^+$, and the forward orbit of $f(\zeta)$ spends a long time in U^- . Hence it inherits the exponential growth of derivatives near the fixed saddle Q . For an integer $i \geq 1$ write $w_i(\zeta) = D_{f(\zeta)} f^{i-1} \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right)$, and define

$$n(\zeta) = \sup \left(\{i > 1 : f^i(\zeta) \in U^-\} \cup \{+\infty\} \right).$$

Then

$$(8) \quad 3^{i-1} < \|w_i(\zeta)\| < 5^{i-1} \quad \text{for every } i \in \{1, 2, \dots, n(\zeta)\}.$$

More precisely, from the bounded distortion near the fixed saddle Q ,

$$(9) \quad \|w_i(\zeta)\| \asymp e^{\lambda^u(\delta_Q)(i-1)} \quad \text{for every } i \in \{1, 2, \dots, n(\zeta)\}.$$

2.6. Binding to critical points. In order to deal with the effect of returns to $I(\delta)$, we now establish a binding argument in the spirit of Benedicks & Carleson [2] which allows one to bind generic orbits which fall inside $I(\delta)$ to suitable critical points, to let it copy the exponential growth along the piece of the critical orbit.

Let ζ be a critical point and let $z \in I(\delta) \setminus \{\zeta\}$. We define a *bound period* $p = p(\zeta, z)$ in the following manner. Consider the leaf $\mathcal{F}^s(f(\zeta))$ of the stable foliation through $f(\zeta)$. This leaf is expressed as a graph of a C^2 function: there exists an open interval J containing 0 and independent of b , and a C^2 function $y \mapsto x^s(y)$ on J such that

$$\mathcal{F}^s(f(\zeta)) = \{(x^s(y), y) : y \in J\}.$$

Choose a small number $\tau > 0$ such that any closed ball of radius $\sqrt{\tau}$ about a point in α_0^- is contained in U^- . For each integer $k \geq 1$ define

$$D_k(\zeta) = \tau \left[\sum_{i=1}^k \frac{\|w_i(\zeta)\|^2}{\|w_{i+1}(\zeta)\|} \right]^{-1}.$$

Write $f(z) = (x_0, y_0)$. If $|x_0 - x^s(y_0)| \leq D_{n(\zeta)}(\zeta)$, then define $p = n(\zeta) + 1$. Otherwise, define p to be the unique integer in $\{2, 3, \dots, n(\zeta)\}$ that satisfies $D_p(\zeta) < |x_0 - x^s(y_0)| \leq D_{p-1}(\zeta)$.

Let S denote the compact lenticular domain bounded by the parabola in $W^s(Q)$ and one of the unstable sides of R (See FIGURE 5). Note that $f(S) \subset U^+$.

Proposition 2.7. *Let γ be a $C^2(b)$ -curve in $I(\delta)$ and ζ a critical point on γ . Let $z \in \gamma \setminus \{\zeta\}$ and $p = p(\zeta, z)$. Then the following holds.*

- (I) *If $p \leq n(\zeta)$, then:*
 - (a) $p \approx -\log |\zeta - z|$;
 - (b) $f^i(z) \in U$ for every $i \in \{1, 2, \dots, p-1\}$;
 - (c) *let v denote any nonzero vector tangent to γ at z . Then $\|D_z f^p(v)\| \geq e^{\lambda_0(p-i)} \|D_z f^i(v)\|$ for every $i \in \{0, 1, \dots, p-1\}$. In particular, $\text{slope}(D_z f^p(v)) \leq \sqrt{b}$;*
- (II) *If $p = n(\zeta) + 1$, then $f^{n(\zeta)}(z) \notin R$.*

A proof of Proposition 2.7 is lengthy. Before entering it we give a couple of remarks and prove one lemma which will be used later.

Remark 2.8. Let $z \in I(\delta) \cap \Lambda$ and suppose that $\text{slope}(E_z^u) \leq \sqrt{b}$. We claim that if there exists a $C^2(b)$ -curve which is tangent to E_z^u and contains a critical point ζ , then $p(\zeta, z) \leq n(\zeta)$ holds. For otherwise $p(\zeta, z) = n(\zeta) + 1$, and Proposition 2.7(II) gives $f^{n(\zeta)}(z) \notin R$. Since $z \in \Lambda \subset \Omega \subset R$ a contradiction arises.

Remark 2.9. As a by-product of the proof of Proposition 2.7 it follows that any $C^2(b)$ -curve in $I(\delta)$ admits at most one critical point.

Let α_1^+ denote the connected component of $R \cap W^s(P)$ containing P , and α_1^- the connected component of $W^s(P) \cap f^{-1}(\alpha_1^+)$ which is not α_1^+ . Let Θ denote the rectangle bordered by α_1^+ , α_1^- and the unstable sides of R . Note that

$$S \subset I(\delta) \subset \Theta.$$

Lemma 2.10. *Let γ be a $C^2(b)$ -curve in $I(\delta)$ which admits a critical point. If $n \geq 1$ is such that $\text{Int}(\Theta) \cap f^i(\gamma) = \emptyset$ for every $i \in \{0, 1, \dots, n-1\}$ and $\text{Int}(\Theta) \cap f^n(\gamma) \neq \emptyset$, then any connected component of $\Theta \cap f^n(\gamma)$ is a $C^2(b)$ -curve.*

Proof. By Lemma 2.3 it suffices to show that for any $z \in \gamma$ with $f^n(z) \in \Theta$ and a nonzero vector v tangent to γ at z , $\|Df^n(v)\| \geq \delta \|Df^i(v)\|$ holds for every $i \in \{0, 1, \dots, n-1\}$. This is a consequence of Lemma 2.2 and Proposition 2.7(I)(c). \square

Proof of Proposition 2.7. We start with establishing three preliminary estimates.

Estimate 1 (horizontal distance). Let $\mathcal{F}^s(f(\zeta)) = \{(x^s(y), y)\}$ denote the leaf of the foliation through $f(\zeta)$ as in Sect.2.5. Write $f(z) = (x_0, y_0)$. In the first step we estimate $|x_0 - x^s(y_0)|$.

Write $f(\zeta) = (x^s(y_1), y_1)$ and $e^s(f(\zeta)) = \begin{pmatrix} \cos \theta(\zeta) \\ \sin \theta(\zeta) \end{pmatrix}$, $\theta(\zeta) \asymp \pi/2$. Define two functions $\xi = \xi(x, y)$ and $\eta = \eta(y)$ implicitly by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^s(y_1) \\ y_1 \end{pmatrix} + \xi \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \eta \cdot \begin{pmatrix} \cos \theta(\zeta) \\ \sin \theta(\zeta) \end{pmatrix}.$$

Solving these equations gives

$$\xi(x, y) = x - x^s(y_1) - \frac{\cos \theta(\zeta)}{\sin \theta(\zeta)}(y - y_1).$$

A direct computation gives

$$\frac{d^2 \xi(x^s(y), y)}{dy^2} = \frac{d\xi}{dx} \frac{d^2 x^s(y)}{dy^2} + \frac{d^2 \xi}{dx^2} \left(\frac{dx^s(y)}{dy} \right)^2 + 2 \frac{d^2 \xi}{dx dy} \frac{dx^s(y)}{dy} + \frac{d^2 \xi}{dy^2}.$$

Using $|\frac{d^2 x^s(y)}{dy^2}| \leq C$ and $|\frac{dx^s(y)}{dy}| \leq C\sqrt{b}$ which follow from conditions (c) (d) in Sect.2.5,

$$\left| \frac{d^2 \xi(x^s(y), y)}{dy^2} \right| \leq C.$$

Since $f(\gamma)$ is tangent to $\mathcal{F}^s(f(\zeta))$ at $f(\zeta)$,

$$\frac{d\xi(x^s(y), y)}{dy}(y_1) = 0.$$

We get

$$(10) \quad |\xi(x^s(y_0), y_0)| = |\xi(x^s(y_0), y_0) - \xi(x^s(y_1), y_1)| \leq C|y_0 - y_1|^2.$$

We also have

$$(11) \quad |y_0 - y_1| \leq |\eta(y_0)|.$$

Parametrize the $C^2(b)$ -curve γ by arc length s so that that $\gamma(s_0) = z$ and $\gamma(s_1) = \zeta$. Then $\zeta - z = \int_{s_0}^{s_1} D_{\gamma(s)} f(\dot{\gamma}(s)) ds$, where the dot “ \cdot ” denotes the s -derivative. Split

$$(12) \quad D_{\gamma(s)} f(\dot{\gamma}(s)) = A(\gamma(s)) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + B(\gamma(s)) \cdot \begin{pmatrix} \cos \theta(\zeta) \\ \sin \theta(\zeta) \end{pmatrix}.$$

The proof of [20, Lemma 2.2] implies

$$|A(\gamma(s))| \asymp 2|\gamma(s) - \zeta| \quad \text{and} \quad |B(\gamma(s))| \leq C\sqrt{b}.$$

Integrations from $s = s_0$ to s_1 gives

$$(13) \quad |\xi(x_0, y_0)| \asymp 2|z - \zeta|^2 \quad \text{and} \quad |\eta(y_0)| \leq C\sqrt{b}|z - \zeta|.$$

Using (10) (11) for $y = y_0$ and the second estimate in (13) we obtain

$$(14) \quad |\xi(x^s(y_0), y_0)| \leq C|y_0 - y_1|^2 \leq C|\eta(y_0)|^2 \leq Cb|\xi(x_0, y_0)|.$$

This yields

$$(15) \quad |x_0 - x^s(y_0)| = |\xi(x_0, y_0) - \xi(x^s(y_0), y_0)| \asymp 2|\zeta - z|^2.$$

This implies that the tangency between $f(\gamma)$ and $\mathcal{F}^s(f(\zeta))$ at $f(\zeta)$ is quadratic. If there were two critical points on γ , then the two leaves through the critical values intersect each other. This is absurd because the leaves of the foliation \mathcal{F}^s are integral curves of C^1 vector fields.

Estimate 2 (slopes and lengths of iterated curves). Let l denote the straight segment connecting $f(z) = (x_0, y_0)$ and $(x^s(y_0), y_0) \in \mathcal{F}^s(f(\zeta))$. Arguing inductively, it is possible to show that for every $i \in \{1, 2, \dots, p-1\}$ the slopes of tangent directions of $f^i(l)$ are everywhere $\leq \sqrt{b}$, and

$$(16) \quad \begin{aligned} \text{length}(f^i(l)) &\asymp \text{length}(l) \|w_{i+1}(\zeta)\| \leq D_{p-1}(\zeta) \|w_{i+1}(\zeta)\| \\ &\leq D_{p-1}(\zeta) \|w_p(\zeta)\| \approx \tau. \end{aligned}$$

The \asymp follows from the bounded distortion near Q , and the first inequality from the definition of the bound period p . The \approx follows from the next estimate: using (9) we have

$$(17) \quad D_k(\zeta) = \tau \left[\sum_{i=1}^k \frac{\|w_i(\zeta)\|^2}{\|w_{i+1}(\zeta)\|} \right]^{-1} \asymp \tau \left[\sum_{i=1}^k e^{\lambda^u(\delta_Q)(i-1)} \right]^{-1} \approx \tau e^{-\lambda^u(\delta_Q)k}.$$

Estimate 3 (length of fold periods). Define a *fold period* $q = q(\zeta, z)$ by

$$(18) \quad q = \min\{i \in \{1, 2, \dots, p-1\} : |\zeta - z|^\beta \|w_{j+1}(\zeta)\| \geq 1 \text{ for every } j \in \{i, i+1, \dots, p-1\}\},$$

where

$$\beta = -\frac{1}{\log b}.$$

This definition makes sense because $|\zeta - p|^\beta \|w_p(\zeta)\| = |\zeta - z|^{\beta-2} |\zeta - z|^2 \|w_p(\zeta)\| > 1$ from (17). By the definition of q and (8) we have

$$1 \leq |\zeta - z|^\beta \|w_{q+1}(\zeta)\| \leq |\zeta - z|^\beta 5^q.$$

This yields

$$(19) \quad q \geq \log |\zeta - z|^{-\frac{\beta}{\log 5}}.$$

Proof of Proposition 2.7 (continued). Recall that any closed ball of radius $\sqrt{\tau}$ about a point in α_0^- is contained in U . Hence, the conditions $f^{n(\zeta)}(\zeta) \in U^-$, $f^{n(\zeta)+1}(\zeta) \notin U^-$ and the hyperbolicity of the fixed saddle Q altogether imply that there is a ball of radius of order $\sqrt{\tau}$ about $f^{n(\zeta)-1}(\zeta)$ which is contained in U . Since $p \leq n(\zeta)$, for every $i \in \{1, 2, \dots, p-1\}$ there is a ball of radius of order $\sqrt{\tau}$ about $f^i(\zeta)$ which is contained in U . Since $\text{length}(f^{i-1}(l)) < \tau$ as above, we obtain $|f^i(\zeta) - f^i(z)| < \text{length}(f^{i-1}(l)) + (Cb)^i \ll \sqrt{\tau}$, and therefore $f^i(z) \in U$ and (a) holds.

Using (15) (16) and (9) we have

$$|\zeta - z|^2 < D_{p-1}(\zeta) \approx \tau e^{-\lambda^u(\delta_Q)p}.$$

Taking logs of both sides and rearranging the result gives $p \leq -\log |\zeta - z|^{\frac{3}{2 \log 2}}$ because $\lambda^u(\delta_Q) \rightarrow \log 4$ as $b \rightarrow 0$. Since $3|\zeta - z|^2 > D_p(\zeta)$, the lower estimate follows similarly and (b) holds.

Write $e^s(f(z)) = \begin{pmatrix} \cos \theta(z) \\ \sin \theta(z) \end{pmatrix}$, $\theta(z) \approx \pi/2$. Recall that v is any nonzero vector tangent to γ at z . Split

$$(20) \quad \frac{1}{\|v\|} \cdot D_z f(v) = A_0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + B_0 \cdot \begin{pmatrix} \cos \theta(z) \\ \sin \theta(z) \end{pmatrix}.$$

From (12) (20) we have

$$\begin{aligned} A(z) + B \cos \theta(\zeta) &= A_0 + B_0 \cos \theta(z); \\ B(z) \sin \theta(\zeta) &= B_0 \sin \theta(z). \end{aligned}$$

Solving these equations gives

$$A_0 - A(z) = B(z)(\cos \theta(\zeta) - \cos \theta(z)) + B(z) \left(1 - \frac{\sin \theta(\zeta)}{\sin \theta(z)}\right) \cos \theta(z).$$

The right-hand-side is $\leq |B(z)| |\zeta - z| \leq C\sqrt{b} |\zeta - z|$ in modulus, and hence we have $|A_0| \asymp 2|\zeta - z|$. Therefore

$$(21) \quad |A_0| \cdot \|D_{f(z)} f^{i-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix}\| \asymp 2|\zeta - z| \cdot \|w_i(\zeta)\| \text{ for every } i \in \{0, 1, \dots, p\}.$$

Let $i \in \{q, q+1, \dots, p\}$. From the definition of $q = q(\zeta, z)$ in (18),

$$(22) \quad |\zeta - z| \cdot \|w_i(\zeta)\| \geq |\zeta - z|^{1-\beta}.$$

On the other hand,

$$\|D_{f(z)}^i(e^s(f(z)))\| \leq (Cb)^i \leq (Cb)^q \leq |\zeta - z|^{\frac{5}{3}}.$$

The last inequality holds for sufficiently small δ , by virtue of the definition of q and its lower bound in (19). Hence

$$(23) \quad \|D_z f^i(v)\| \asymp 2|\zeta - z| \cdot \|w_i(\zeta)\| \cdot \|v\| \quad \text{for every } i \in \{q, q+1, \dots, p\}.$$

This yields

$$\frac{\|D_z f^p(v)\|}{\|D_z f^i(v)\|} \asymp \frac{\|w_p(\zeta)\|}{\|w_i(\zeta)\|} \asymp e^{\lambda^u(\delta_Q)(p-i)},$$

and therefore $\|D_z f^p(v)\| \geq e^{\lambda_0(p-i)} \|D_z f^i(v)\|$.

Using (23) with $i = p$ and then (17) gives

$$(24) \quad \|D_z f^p(v)\| \asymp 2|\zeta - z| \cdot \|w_p(\zeta)\| \cdot \|v\| \approx \tau^2 |\zeta - z|^{-1} \|v\| \approx \tau^{\frac{3}{2}} e^{\frac{p}{2} \lambda^u(\delta_Q)} \|v\|,$$

and therefore $\|D_z f^p(v)\| \geq e^{\lambda_0 p} \|v\|$ provided δ is sufficiently small and hence p is large.

We now treat the case $i \in \{1, 2, \dots, q-1\}$. Using $\|w_i(\zeta)\| < \|w_q(\zeta)\|$ and the definition of q we have

$$|\zeta - z| \cdot \|w_i(\zeta)\| \leq |\zeta - z| \cdot \|w_q(\zeta)\| < |\zeta - z|^{1-\beta} < \sqrt{\delta}.$$

For the other component in the splitting,

$$\|D_{f(z)}^i(e^s(f(z)))\| \leq (Cb)^i \leq Cb.$$

Hence $\|D_z f^i(v)\| < \|v\|$. Using this and (24) we obtain

$$\|D_z f^p(v)\| > \frac{\|D_z f^i(v)\|}{\|v\|} \|D_z f^p(v)\| \asymp \|D_z f^i(v)\| |\zeta - z|^{-1} \approx \frac{1}{\sqrt{\tau}} \|D_z f^i(v)\| e^{\frac{\lambda^u(\delta_Q)}{2} p}.$$

This yields $\|D_z f^p(v)\| \geq e^{\lambda_0(p-i)} \|D_z f^i(v)\|$ provided δ is sufficiently small. We have proved (c).

It is left to prove (II). By the definition of $n(\zeta)$ we have $f^{n(\zeta)}(\zeta) \in U^-$ and $f^{n(\zeta)+1}(\zeta) \notin U^-$. This and the choice of τ together imply that there is a closed ball of radius of order $\sqrt{\tau}$ about $f^{n(\zeta)}(\zeta)$ which does not intersect α_0^- . Since $\zeta \in S$, $f^{n(\zeta)}(\zeta)$ is at the left of α_0^- . Since $\text{length}(f^{n(\zeta)-1}(l)) < \tau$ we have $|f^{n(\zeta)}(\zeta) - f^{n(\zeta)}(z)| < \text{length}(f^{n(\zeta)-1}(l)) + (Cb)^n \ll \sqrt{\tau}$. This implies $f^{n(\zeta)}(z) \notin R$. \square

3. GLOBAL CONSTRUCTION

In this section we use the results in Sect.2 to construct an induced system with uniformly hyperbolic behavior. From the induced system we extract a hyperbolic set, the dynamics on which is conjugate to the full shift on a finite number of symbols. This hyperbolic set will be used to complete the proof of the main theorem in the next section.

3.1. Construction of induced system. In this subsection we deliberately construct an induced system with uniformly hyperbolic Markov structure, with countably infinite number of branches. Although a similar construction has been done in [23] at $a = a^*$ to analyze equilibrium measures for the potential $-t \log J^u$ as $t \rightarrow +\infty$, it does not fit to our purpose of studying the case $t \rightarrow -\infty$. Moreover, a treatment of the case $a^{**} < a < a^*$ brings additional difficulties which are not present in [23]. We exploit the geometric structure of invariant manifolds of P and Q which are “not destroyed yet” by the homoclinic bifurcation.

We say a $C^2(b)$ -curve γ in R *stretches across* Θ if both endpoints of γ are contained in the stable sides of Θ . Let ω, ω' be two rectangles in Θ such that $\omega \subset \omega'$. We call ω a *u-subrectangle* of ω' if each stable side of ω' contains one stable side of ω . Similarly, we call ω an *s-subrectangle* of ω' if each unstable side of ω' contains one unstable side of ω .

Proposition 3.1. *There exist a u-subrectangle Θ' of Θ , a large integer $k_0 \geq 1$, a constant $C \in (0, 1)$ and a countably infinite family $\{\omega_k\}_{k \geq k_0}$ of s-subrectangles of Θ' with the following properties:*

- (a) *the unstable sides of Θ' are $C^2(b)$ -curves stretching across Θ and intersecting $\text{Int}(S)$;*
- (b) *for every nonzero vector v tangent to the unstable side of Θ' and every integer $n > 0$,*
 $\|Df^{-n}(v)\| \leq e^{-\lambda_0 n} \|v\|$;
- (c) *for each ω_k , $\text{Int}(\Theta) \cap f^i(\omega_k) = \emptyset$ for every $i \in \{1, 2, \dots, k-1\}$, and $f^k(\omega_k)$ is a u-subrectangle of Θ' whose unstable sides are $C^2(b)$ -curves stretching across Θ' ;*
- (d) *if $z \in \omega_k \cap \Omega$ and $\text{slope}(E_z^u) \leq \sqrt{b}$, then $\|Df^k|E_z^u\| \geq Ce^{\lambda^u(\delta_Q)k}$.*

Proof. Since the construction is involved, we start with giving a brief sketch. Let γ_0 denote the $C^2(b)$ -curve in W^u which stretches across Θ and is part of the boundary of S . This curve, which obviously satisfies the exponential backward contraction property as in item (b), will be one of the unstable sides of Θ' . In Step 1 we find another $C^2(b)$ -curve stretching across Θ , which will be the other unstable side of Θ' . In Step 2 we construct the rectangles $\{\omega_k\}_{k \geq k_0}$ by subdividing Θ' into smaller rectangles, with a family of compact curves in $W^s(P)$. Both of the steps depend on the orientation of the map f .

Step 1 (construction of Θ'). Suppose that α, α' are two compact curves in $R \cap W^s(P)$ which join the two unstable sides of R and intersect γ_0 exactly at one point. We write $\alpha \prec \alpha'$ if $\text{proj}(\alpha \cap \gamma_0) < \text{proj}(\alpha' \cap \gamma_0)$, where proj denotes the projection to the first coordinate.

Set $\tilde{\alpha}_0 = \alpha_1^+$ and $\tilde{\alpha}_1 = \alpha_1^-$. Let $\{\tilde{\alpha}_k\}_{k \geq 0}$ denote the sequence of compact curves in $R \cap W^s(P)$ with the following properties: each $\tilde{\alpha}_k$ joins the two unstable sides of R ; $\tilde{\alpha}_k \prec \tilde{\alpha}_{k-1}$ and $f(\tilde{\alpha}_k) \subset \tilde{\alpha}_{k-1}$ for every $k \in \{1, 2, \dots\}$. Notice that $\tilde{\alpha}_k$ converges to α_0^- as $k \rightarrow \infty$.

For every $k \geq 1$ the set $R \cap f^{-2}(\tilde{\alpha}_{k-1})$ has three or four connected components. Two of them are $\tilde{\alpha}_{k+1}$ and the connected component of $R \cap f^{-1}(\tilde{\alpha}_k)$ which is not $\tilde{\alpha}_{k+1}$. Let α_k denote the union of the remaining one or two connected components of $R \cap f^{-2}(\tilde{\alpha}_{k-1})$. By definition, α_k is located near the origin (If $a = a^*$, then for every $k \geq 0$, α_k has two connected components. If $a^{**} < a < a^*$ then there exists an integer $k' = k'(a)$ such that α_k has two connected components if and only if $k < k'$). Choose a large integer $\hat{k} \geq 1$ such that $\alpha_k \subset I(\delta)$ holds for every $k \geq \hat{k}$.

The rest of the construction of Θ' depends on the orientation of f . We first consider the case $\det Df > 0$. Let $k \geq \hat{k}$. The set α_k intersects γ_0 exactly at two points. Let γ_k^-, γ_k^+ denote the compact curve in γ_0 whose endpoints are in α_k and α_{k+1} , and satisfy $\sup \text{proj}(\gamma_k^-) < \inf \text{proj}(\gamma_k^+)$. Since $\text{Int}(\Theta') \cap f^i(\gamma_k^k) = \emptyset$ for every $i \in \{1, 2, \dots, k-1\}$ and

$f^k(\gamma^k) \subset \Theta$, $f^k(\gamma^k)$ is a $C^2(b)$ -curve from Lemma 2.10. In addition, since its endpoints are contained in the stable sides of Θ' , $f^k(\gamma^k)$ stretches across Θ . Enlarging \hat{k} if necessary, we have $\text{Int}(S) \cap f^k(\gamma_k^+) \neq \emptyset$ for every $k \geq \hat{k}$. Define Θ' to be the rectangle bordered by γ_0 , $f^{\hat{k}}(\gamma_{\hat{k}}^+)$ and the stable sides of Θ . The exponential backward contraction property in item (b) may be proved along the line of the proof of (25) and hence we omit it.

Remark 3.2. There is no particular reason for our choice of γ_k^+ . Choosing γ_k^- does the same job.

The case $\det Df < 0$ is easier to handle. Since $a \in (a^{**}, a^*] \subset \mathcal{G}$, there is the unique compact domain bounded by $f^{-2}(W_{\text{loc}}^s(Q))$ and ℓ^u , which is contained in S . Define Θ' denote the rectangle bordered by the stable sides of Θ , γ_0 and the $C^2(b)$ -curve in $W^u(Q)$ which stretches across Θ and contains ℓ^u . Items (a) and (b) in Proposition 3.1 hold.

Step 2 (construction of ω_k). The set $\alpha_k \cap \Theta'$ consists of two connected components, one at the left of S and the other at the right of S . Let $\hat{\omega}_k$ denote the connected component of $\alpha_k \cap \Theta'$ which lies at the right of S . Then $f^k(\hat{\omega}_k)$ is a u -subrectangle of Θ , whose unstable sides are $C^2(b)$ -curves stretching across Θ .

In the case $\det Df > 0$, choose a sufficiently large integer $k_0 > \hat{k}$ depending on the parameter a such that for every $k \geq k_0$, $f^k(\hat{\omega}_k)$ is a u -subrectangle of Θ' . Set $\omega_k = \hat{\omega}_k$. In the case $\det Df < 0$, $f^k(\hat{\omega}_k)$ may not be contained in Θ' (See FIGURE 6), and this is always the case for $a < a^*$ and sufficiently large k . However, note that $f^{k+2}(\hat{\omega}_k)$ contains a unique u -subrectangle ω' of Θ' . Set $k_0 = \hat{k} + 2$ and $\omega_k = f^{-k}(\hat{\omega}_{k-2})$ for every $k \geq k_0$. This finishes the construction of $\{\omega_k\}_{k \geq k_0}$. Item (c) is a direct consequence of the construction.

To prove item (d) we need the next uniform upper bound on the length of bound periods.

Lemma 3.3. *There is a constant $E > 0$ such that if $z \in \bigcup_{k \geq k_0} \omega_k$ and ζ is a critical point on a $C^2(b)$ -curve which is tangent to v , then $p(\zeta, z) \leq E$.*

Proof. Let z, ζ be as in the statement of the lemma and assume $z \in \omega_k$. By construction, one of the unstable sides of ω_k is contained in the $C^2(b)$ -curve γ_0 which is not contained in the unstable sides of Θ and stretches across Θ . Let ζ' denote the critical point on γ_0 . With a slight abuse of notation, let $\mathcal{F}^s(\alpha_0^+)$ denote the leaf containing α_0^+ . The leaf $\mathcal{F}^s(f(\zeta'))$ lies at the right of $\mathcal{F}^s(\alpha_0^+)$, and $\mathcal{F}^s(f(\zeta))$ lies at the right of $\mathcal{F}^s(\alpha_0^+)$. Since $f(z) \in R$ we have

$$3|\zeta - z|^2 \geq \inf\{|z_1 - z_2| : z_1 \in \mathcal{F}^s(\alpha_0^+), z_2 \in \mathcal{F}^s(f(\zeta'))\} > 0.$$

Taking logs of both sides and then using Proposition 2.7(a) yields the claim. \square

Since the point z is sandwiched by the two $C^2(b)$ -curves intersecting $\text{Int}(\Theta)$, there exists a $C^2(b)$ -curve which is tangent to E_z^u and contains a critical point ζ . Let $p = p(\zeta, z)$ denote the bound period given by Proposition 2.7. Since $\text{slope}(E_{f^p(z)}^u) \leq \sqrt{b}$, $f^n(z) \notin \text{Int}(\Theta)$ for every $n \in \{n, n+1, \dots, k-1\}$ and $f^k(z) \in \Theta$, the bounded distortion for iterates near Q gives

$$\|D_{f^p(z)} f^{k-p} | E_{f^p(z)}^u\| \approx e^{\lambda^u(\delta_Q)(k-p)}.$$

Using $\|Df^p | E_z^u\| > 1$ and $p \leq E$ by Lemma 3.3 we obtain

$$\|Df^k | E_z^u\| = \|Df^p | E_z^u\| \cdot \|Df^{k-p} | E_{f^p(z)}^u\| \geq C e^{-\lambda^u(\delta_Q)E} \|D_{f^p(z)} f^{k-p} | E_{f^p(z)}^u\|,$$

and hence item (d) holds. This completes the proof of Proposition 3.1. \square

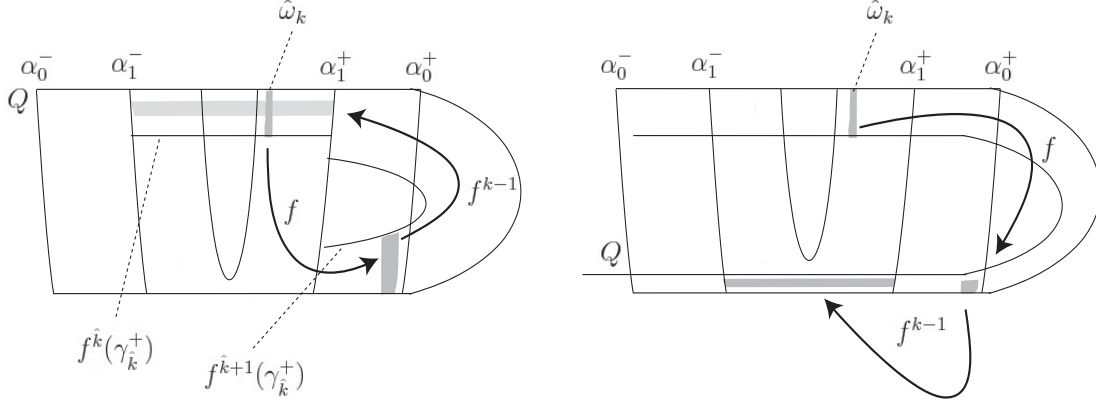


FIGURE 6. The rectangles $\hat{\omega}_k, f(\hat{\omega}_k), f^k(\hat{\omega}_k)$ (shaded) for $a \in (a^{**}, a^*)$: the case $\det Df > 0$ (left); the case $\det Df < 0$ (right).

3.2. Symbolic dynamics. From the induced system in Proposition 3.1 we extract a finite number of branches, and construct a conjugacy to the full shift on a finite number of symbols. For two positive integers q_0, q_1 with $q_0 < q_1$ define

$$\Sigma(q_0, q_1) = \{\underline{a} = \{a_i\}_{i \in \mathbb{Z}} : a_i \in \{q_0, q_0 + 1, \dots, q_1\}\}.$$

This is the set of two-sided sequences with $q_1 - q_0 + 1$ -symbols. Endow $\Sigma(q_0, q_1)$ with the product topology of the discrete topology of $\{q_0, q_0 + 1, \dots, q_1\}$.

Proposition 3.4. *For all integers $1 \leq q_0 < q_1$ there exist a continuous injection $\pi : \Sigma(q_0, q_1) \rightarrow \Omega$ and a constant $C \in (0, 1)$ such that the following holds:*

- (a) for every $\underline{a} \in \Sigma(q_0, q_1)$, $\text{slope}(E_{\pi(\underline{a})}^u) \leq \sqrt{b}$;
- (b) for every $\underline{a} = \{a_i\}_{i \in \mathbb{Z}} \in \Sigma(q_0, q_1)$, $\|Df^{a_0}|E_{\pi(\underline{a})}^u\| \geq Ce^{\lambda^u(\delta_Q)a_0}$ and $\|Df^{a_i}|E_{f^{a_0+a_1+\dots+a_{i-1}}(\pi(\underline{a}))}^u\| \geq Ce^{\lambda^u(\delta_Q)a_i}$ for every integer $i \geq 1$;
- (c) $\underline{a} \in \Sigma(q_0, q_1) \mapsto E_{\pi(\underline{a})}^u$ is continuous.

Proof. Let $\underline{a} = \{a_i\}_{i \in \mathbb{Z}} \in \Sigma(q_0, q_1)$. For each integer $j \geq 1$ define

$$\omega_j^s = \omega_{a_0} \cap \left(\bigcap_{i=1}^j f^{-a_0} \circ f^{-a_1} \circ \dots \circ f^{-a_{i-1}}(\omega_{a_i}) \right)$$

and

$$\omega_j^u = \bigcap_{i=1}^j f^{a_{-1}} \circ f^{a_{-2}} \circ \dots \circ f^{a_{-i}}(\omega_{a_{-i}}).$$

Note that $\{\omega_j^s\}_{j \geq 1}$ is a decreasing sequence of s -subrectangles of Θ' , and $\{\omega_j^u\}_{j \geq 1}$ is a decreasing sequence of u -subrectangles of Θ' . Define a coding map $\pi : \Sigma(q_0, q_1) \rightarrow \Omega$ by

$$\{\pi(\underline{a})\} = \left(\bigcap_{j=1}^{+\infty} \omega_j^s \right) \cap \left(\bigcap_{j=1}^{+\infty} \omega_j^u \right).$$

We show below that the right-hand-side is a singleton, and so π is well-defined.

By Corollary 2.6 and the fact that f contracts area, the set $\bigcap_{j=1}^{+\infty} \omega_j^u$ is a C^1 curve which connects the stable sides of ω_{a_0} . By Corollary 2.6 again, for any nonzero vector v tangent to $\bigcap_{j=1}^{+\infty} \omega_j^u$ there exists a $C^2(b)$ -curve which is tangent to v and contains a critical point. By Proposition 2.7(b) and Lemma 2.2, for every integer $m \geq 1$ we have

$$\text{length} \left(\left(\bigcap_{j=1}^m \omega_j^s \right) \cap \left(\bigcap_{j=1}^{+\infty} \omega_j^u \right) \right) \leq \exp \left(-\lambda_0 \sum_{i=0}^m a_i \right).$$

The right-hand side goes to 0 as $m \rightarrow \infty$. This means that π is well-defined.

The continuity of π is obvious. To show the injectivity, assume $\underline{a} \neq \underline{a}'$ and $\pi(\underline{a}) = \pi(\underline{a}')$. Then there exists an integer i such that $f^i(\pi(\underline{a}))$ belongs to two rectangles in $\{\omega_k\}_{k=q_0}^{q_1}$, namely belongs to a curve in $\bigcup_{k \geq k_0} \alpha_k$ which is a stable side of two neighboring rectangles. It follows that $f^{a_0+a_1+\dots+a_i}(\pi(\underline{a})) \in \alpha_1^+$ holds for all large integer $i > 0$. On the other hand, the definition of π gives $f^{a_0+a_1+\dots+a_i}(\pi(\underline{a})) \in \omega_{a_i}$. Since $\omega_{a_i} \cap \alpha_1^+ = \emptyset$ by construction, we obtain a contradiction.

Recall that the unstable direction is characterized by the exponential backward contraction property (2). By Corollary 2.6 and the fact that the C^1 -curve $\bigcap_{j=1}^{+\infty} \omega_j^u$ is obtained as the C^1 -limit of the unstable sides of $\{\omega_j^u\}_{j \geq 1}$. To prove items (a) and (c) it suffices to show that for every integer $j \geq 1$, any unstable side γ of ω_j^u , every $z \in \gamma$, every integer $n \geq 0$ and every vector v tangent to $f^{-n}(\gamma)$ at $f^{-n}(z)$,

$$(25) \quad \|D_{f^{-n}(z)} f^n(v)\| \geq e^{\lambda_0 n} \|v\|.$$

Then Item (b) is a consequence of the construction of π and Proposition 3.1(d).

It is left to prove (25). For each $i \in \{1, 2, \dots, j\}$ define an integer $n_i \leq 0$ by $f^{n_i}(\gamma) \in \omega_{a_i}$. Note that $n_1 < n_2 < \dots < n_{j-1} < n_j = 0$. Below we treat four cases separately.

Case I: $-n = n_i$ for some i . We split the time interval $[n_i, 0]$ into subintervals $[n_l, n_{l+1}]$ ($l = i, i+1, \dots, j-1$). Then we apply the derivative estimates in Lemma 2.2 and Proposition 2.7(c) to $[n_l, n_l + p_l]$ and $[n_l + p_l, n_{l+1}]$ respectively. Recall that $\lambda^u(\delta_Q) \rightarrow \log 4$ as $b \rightarrow 0$ and $\lambda_0 = \frac{99}{100} \log 2$. We obtain

$$(26) \quad \|D_{f^{-n}(z)} f^n(v)\| \geq \left(\prod_{l=i}^{j-1} e^{\lambda_0(n_{l+1}-n_l)} \right) \|v\| = e^{\lambda_0(n_j-n_i)} \|v\| = e^{\lambda_0 n} \|v\|.$$

Case II: $n_i + p_i \leq -n < n_{i+1}$ for some i . Since $f^{n_{i+1}}(z) \in I(\delta)$, Lemma 2.2 gives

$$\|D_{f^{-n}(z)} f^{n_{i+1}+n}(v)\| \geq e^{\lambda_0(n_{i+1}+n)} \|v\|.$$

Using this and (26) with $-n = n_{i+1}$ we get

$$\|D_{f^{-n}(z)} f^n(v)\| = \frac{\|D_{f^{n_{i+1}}(z)} f^{-n_{i+1}}(D_{f^{-n}(z)} f^{n_{i+1}+n}(v))\|}{\|D_{f^{-n}(z)} f^{n_{i+1}+n}(v)\|} \cdot \|D_{f^{-n}(z)} f^{n_{i+1}+n}(v)\| \geq e^{\lambda_0 n} \|v\|.$$

Case III: $n_i < -n < n_i + p_i$. Proposition 2.7(c) gives

$$\|D_{f^{-n}(z)} f^{n_i+p_i+n}(v)\| \geq e^{\lambda_0(n_i+p_i+n)} \|v\|.$$

Combining this with the result in Case II yields the desired inequality.

Case IV: $-n < n_1$. Since $f^{n_1}(\gamma)$ is contained in the unstable sides of Θ' , Proposition 3.1(b) gives $\|D_{f^{-n}(z)}f^{n_1+n}(v)\| \geq e^{\lambda_0(n_1+n)}\|v\|$. From this and (26) with $-n = n_1$ we get the desired inequality. \square

4. PROOF OF THE MAIN THEOREM

In this section we use the results in Sect.2 and finish the proof of the main theorem. Finally we provide more details on the main theorem, on the abundance of parameters beyond a^* satisfying $\mathcal{M}_0(f) = \mathcal{M}(f)$.

4.1. Proof of the main theorem. By virtue of Lemma 2.4, the removability in the main theorem follows from the next

Proposition 4.1. *For any $t < 0$ there exists a measure $\mu \in \mathcal{M}(f)$ such that*

$$h_\mu(f) - t\lambda^u(\mu) > -t\lambda^u(\delta_Q).$$

Proof. Let $q > 0$ be the square of a large integer, to be determined later depending on t . Set $\Sigma(q) = \Sigma(q - \sqrt{q} + 1, q)$, and let $\sigma: \Sigma(q) \rightarrow \Sigma(q)$ denote the left shift. For each $\underline{a} \in \Sigma(q)$ define $r(\underline{a}) = \sum_{i=0}^{q-1} a_i$. Given a σ^q -invariant Borel probability measure μ , define a Borel measure $\mathcal{L}(\mu)$ by

$$\mathcal{L}(\mu) = \frac{1}{\int r d\mu} \sum_{[a_0, a_1, \dots, a_{q-1}]} \sum_{i=0}^{a_0+a_1+\dots+a_{q-1}-1} f_*^i(\pi_* (\mu|_{[a_0, a_1, \dots, a_{q-1}]})),$$

where $[a_0, a_1, \dots, a_{q-1}] = \{\underline{b} \in \Sigma(q) : b_i = a_i, i \in \{0, 1, \dots, q-1\}\}$ and $\pi: \Sigma(q) \rightarrow \Omega$ is the coding map given by Proposition 3.1. Then $\mathcal{L}(\mu)$ is an f -invariant and a probability. For $t < 0$ define $\Phi_t: \Sigma(q) \rightarrow \mathbb{R}$ by

$$\Phi_t(\underline{a}) = -t \log \|Df^{r(\underline{a})}|E_{\pi(\underline{a})}^u\|.$$

From Proposition 3.4, Φ_t is continuous and satisfies

$$\Phi_t(\underline{a}) \geq -tq \log C - tr(\underline{a})\lambda^u(\delta_Q).$$

Since $q^2/2 \leq r(\underline{a}) \leq q^2$ and $0 < C < 1$ we have

$$\frac{\Phi_t(\underline{a})}{r(\underline{a})} \geq \frac{-2t \log C}{q} - t\lambda^u(\delta_Q).$$

Let μ_0 denote the measure of maximal entropy of σ^q . For each integer $n \geq 1$ set $P_n = \{\underline{a} \in \Sigma(q) : \sigma^{qn}(\underline{a}) = \underline{a}\}$. Since r and Φ_t are continuous, as $n \rightarrow \infty$ we have

$$\frac{1}{\#P_n} \sum_{\underline{a} \in P_n} r(\underline{a}) \rightarrow \int r d\mu_0 \quad \text{and} \quad \frac{1}{\#P_n} \sum_{\underline{a} \in P_n} \Phi_t(\underline{a}) \rightarrow \int \Phi_t d\mu_0.$$

It follows that

$$\frac{\int \Phi_t d\mu_0}{\int r d\mu_0} \geq \frac{-2t \log C}{q} - t\lambda^u(\delta_Q),$$

for otherwise we would obtain a contradiction. Since the entropy of (σ^q, μ_0) is $q \log \sqrt{q}$ and $\int r d\mu_0 \leq q^2$ we obtain

$$\begin{aligned} P(-t \log J^u) &\geq h(\mathcal{L}(\mu_0)) - t \int \log J^u d\mathcal{L}(\mu_0) = \frac{1}{\int r d\mu_0} \left(q \log \sqrt{q} + \int \Phi_t d\mu_0 \right) \\ &\geq \frac{1}{q} \log \sqrt{q} - \frac{2t \log C}{q} - t\lambda^u(\delta_Q) > -t\lambda^u(\delta_Q). \end{aligned}$$

The strict inequality in the last line holds provided $q > e^{4t \log C}$. \square

Remark 4.2. It is not hard to show that the set $\pi(\Sigma(q))$ is a hyperbolic set. However we do not need this fact.

To finish, it is left to show $P(-t \log J^u) = -t\lambda_M^u + o(1)$ as $t \rightarrow -\infty$ provided $a = a^*$. According to [23] let us call a measure $\mu \in \mathcal{M}(f)$ a $(-)$ -ground state if there exists a sequence $\{t_n\}_n$, $t_n \searrow -\infty$ such that μ_{t_n} is an equilibrium measure for $-t_n \log J^u$ and μ_{t_n} converges weakly to μ as $n \rightarrow \infty$. If $P(-t \log J^u) + \lambda_M^u t \not\rightarrow 0$ as $t \rightarrow -\infty$, then the upper semi-continuity of entropy (see [17]) would imply the existence of a $(-)$ -ground state with positive entropy. If $\det Df > 0$, we obtain a contradiction to [23, Theorem A(b)] which states that the Dirac measure at Q is the unique $(-)$ -ground state. Even if $\det Df < 0$, the proof of [23, Theorem A(b)] works and we obtain the same contradiction. This completes the proof of the main theorem. \square

4.2. Abundance of parameters satisfying $\mathcal{M}_0(f) = \mathcal{M}(f)$. In the main theorem we have reduced the class of measures to consider: only those measures which give full weight to the Borel set Λ were taken into consideration. To claim that $\mathcal{M}_0(f) = \mathcal{M}(f)$ holds for many parameters except a^* , some preliminary discussions are necessary.

From the Oseledec theorem [14] and the two-dimensionality of the system, one of the following holds for each measure $\mu \in \mathcal{M}(f)$ which is ergodic:

- (a) there exist a real number $\chi(\mu)$ such that for μ -a.e. $z \in \Omega$ and for any vector $v \in T_z \mathbb{R}^2 \setminus \{0\}$,

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|Df^n(v)\| = \chi(\mu) \text{ and } \int \log |\det Df| d\mu = 2\chi(\mu);$$

- (b) there exist two real numbers $\chi^1(\mu) < \chi^2(\mu)$ and for μ -a.e. $z \in \Omega$ a non-trivial splitting $T_z \mathbb{R}^2 = E_z^1 \oplus E_z^2$ such that for any vector $v^i \in E_z^i \setminus \{0\}$ ($i = 1, 2$),

$$\lim_{n \rightarrow \pm\infty} \frac{1}{n} \log \|Df^n(v^i)\| = \chi^i(\mu) \text{ (} i = 1, 2 \text{) and } \int \log |\det Df| d\mu = \chi^1(\mu) + \chi^2(\mu).$$

We say μ is a *hyperbolic measure* if (b) holds and $\chi^1(\mu) < 0 < \chi^2(\mu)$.

Lemma 4.3. *Every f -invariant ergodic Borel probability measure is a hyperbolic measure if and only if $\mathcal{M}_0(f) = \mathcal{M}(f)$.*

Proof. Let $\mu \in \mathcal{M}(f)$ be ergodic. Then $\mu(\Lambda) = 1$ if and only if μ is a hyperbolic measure, see e.g., [9]. The “if” part follows from this. The “only if” part is a consequence of the ergodic decomposition of invariant Borel probability measures. \square

It was proved in [21, 22] that if additionally $\{f_a\}$ is C^4 in a, x, y , then for sufficiently small $b > 0$ there exists a set Δ of a -values in $(a^{**}, a^*]$ containing a^* with the following properties:

- $\lim_{\varepsilon \rightarrow +0} (1/\varepsilon) \text{Leb}(\Delta \cap [a^* - \varepsilon, a^*]) = 1$, where $\text{Leb}(\cdot)$ denotes the one-dimensional Lebesgue measure;
- if $a \in \Delta$, then the Lebesgue measure of the set $\{z \in \mathbb{R}^2: \{f^n(z)\}_{n \in \mathbb{N}} \text{ is bounded}\}$ is zero;
- if $a \in \Delta$, then any ergodic measure is a hyperbolic measure.

In other words, the dynamics for parameters in Δ is like Smale's horseshoe. However, whether or not the dynamics is uniformly hyperbolic for $a \in \Delta \setminus \{a^*\}$ is a wide open problem. We even do not know if there exists an increasing sequence of uniformly hyperbolic parameters in Δ converging to a^* .

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